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LETTER TO THE EDITOR

Painlevé analysis, Yoshida's theorems and direct methods in the search for integrable Hamiltonians

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Abstract. Treating as an example two-dimensional similarity invariant Hamiltonians with polynomial velocity-dependent potentials, we show how Painlevé analysis, Yoshida's theorems and direct methods can be combined in the search for integrable Hamiltonians. New integrable systems are found, one of which has a polynomial second invariant of fifth order in the momenta. Most of the necessary calculations have been performed by applying various REDUCE packages developed for that purpose.

Integrability of low-dimensional Hamiltonian and non-Hamiltonian systems has been a very active field of research in the last few years, the reasons being, on the one hand, the success of singular point analysis as a method to find candidates for integrable systems [1-5] and, on the other hand, the availability of algebraic computing programs which make the extensive calculations necessary to perform this analysis and to find constants of the motion possible [6-8].

Different forms of singular point analysis have been used in the literature. There is Painlevé analysis, as proposed by Ablowitz *et al* [1] and afterwards used and refined by many authors [2, 3, 5] and there is Yoshida's method [4] which has been applied in [9, 10]. In the way they have been applied the advantages of these two approaches are that they restrict strongly the number of candidates for integrability and can provide information on the form of possible constants of the motion, respectively. It is the aim of this letter to show how these advantages can be combined. The relation between the 'resonances' used in Painlevé analysis and the 'Kowalevski exponents' (KE) introduced by Yoshida will be given. Then it will be possible to show how the information on the form of the constants of the motion contained in the KE can also be found in the resonances or vice versa, and how the restrictions imposed on the resonances by the Painlevé conjecture can be translated in restrictions on the KE.

For concreteness we consider two-dimensional Hamiltonian systems with polynomial velocity-dependent potential, defined by

$$H(p, q)=\tfrac{1}{2}p_1^2+\tfrac{1}{2}p_2^2+p_1A(q_1, q_2)+p_2B(q_1, q_2)+C(q_1, q_2). \qquad (1)$$

Known integrable Hamiltonians of this form and the direct methods to find the second invariant have been reviewed recently by Hietarinta [11] (see also [12, 13]).

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Yoshida's method can be applied when A and B in (1) are homogeneous of degree d and C is homogeneous of degree $2d$ since then the Hamilton equations of motion are invariant under the similarity transformation

$$t \to \alpha^{-1} t \qquad q_i \to \alpha^{g} q_i \qquad p_i \to \alpha^{g'} p_i \qquad i = 1, 2 \tag{2}$$

the weighted degree of q and p respectively being $g = 1/(d-1)$ and $g' = d/(d-1)$. Kowalevski exponents are associated with the variational problem around special solutions of the Hamilton equations of motion given by

$$q_i = c_i (t - t_0)^{-g} \qquad i = 1, 2 \tag{3a}$$

$$p_i = c_i' (t - t_0)^{-g'} \qquad i = 1, 2 \tag{3b}$$

with c_i and c_i' solutions of

$$-g c_i = \partial H / \partial p_i |_{q = c, p = c'} \tag{4a}$$

$$-g' c_i' = -\partial H / \partial q_i |_{q = c, p = c'}. \tag{4b}$$

Equivalently, one may consider the variational problem around special solutions of the form (3a) of the Euler–Lagrange equations ($W_{,1} \equiv \partial W / \partial q_1$, etc):

$$\ddot{q}_1 = \dot{q}_2 U(q_1, q_2) - W_{,1}(q_1, q_2) \tag{5a}$$

$$\ddot{q}_2 = -\dot{q}_1 U(q_1, q_2) W_{,2}(q_1, q_2) \tag{5b}$$

where

$$U(q_1, q_2) = A_{,2}(q_1, q_2) - B_{,1}(q_1, q_2) \tag{6}$$

$$W(q_1, q_2) = C(q_1, q_2) - \tfrac{1}{2} A^2(q_1, q_2) - \tfrac{1}{2} B^2(q_1, q_2) \tag{7}$$

with the parameters c_i, $i = 1, 2$, solutions of

$$F_1(q_1, q_2) \equiv -g(g+1) c_1 - g c_2 U(c_1, c_2) - W_{,1}(c_1, c_2) = 0 \tag{8a}$$

$$F_2(q_1, q_2) \equiv -g(g+1) c_2 + g c_1 U(c_1, c_2) - W_{,2}(c_1, c_2) = 0. \tag{8b}$$

Equations (8) are obtained from equations (4) through elimination of c_i', $i = 1, 2$. Substituting

$$q_i = c_i (t - t_0)^{-g} + d_i (t - t_0)^{-g + \rho} \qquad i = 1, 2 \tag{9}$$

with c_i a solution of (8), into the linearised form of (5) one obtains a pair of linear equations for d_i, $i = 1, 2$, the determinant of which is the Kowalevski determinant $K(\rho)$ given by

$$K(\rho) = M_{11} M_{22} - M_{12} M_{21} \tag{10}$$

with

$$M_{11}(\rho; c_1, c_2) \equiv -(\rho - g)(\rho - g - 1) - g c_2 U_{,1}(c_1, c_2) - W_{,11}(c_1, c_2) \tag{11a}$$

$$M_{12}(\rho; c_1, c_2) \equiv (\rho - g) U(c_1, c_2) - g c_2 U_{,2}(c_1, c_2) - W_{,12}(c_1, c_2) \tag{11b}$$

$$M_{21}(\rho; c_1, c_2) \equiv -(\rho - g) U(c_1, c_2) + g c_1 U_{,1}(c_1, c_2) - W_{,12}(c_1, c_2) \tag{11c}$$

$$M_{22}(\rho; c_1, c_2) \equiv -(\rho - g)(\rho - g - 1) + g c_1 U_{,2}(c_1, c_2) - W_{,22}(c_1, c_2). \tag{11d}$$

For every solution of (8) the roots of $K(\rho)$ are a set of KE. In Hamiltonian systems these exponents come in pairs $(\bar{\rho}, g_H - 1 - \bar{\rho})$ with g_H the weighted degree of the Hamiltonian and the pair $(-1, g_H)$ is always present [4].

Therefore here all sets are of the form

$$\rho = -1, g_H, \bar{\rho}, g_H - 1 - \bar{\rho} \tag{12}$$

with $g_H = 2d/(d-1)$.

Yoshida has shown [4] that if there exists a second invariant $L(p, q)$ of weighted degree g_L satisfying the condition that its gradient does not vanish at $q = c$, $p = c'$, with c, c' a solution of (4), then a KE $\bar{\rho} = g_L$ is associated with this c, c'. (In the case $g_L = g_H$ there is the supplementary condition that the gradients of L and H must be linearly independent). Yoshida has also shown [4] that a necessary condition for the existence of an algebraic second invariant is that all KE are rational numbers. From this one can learn that a direct search for a rational second invariant should only be done when all KE are rational and that a second invariant should first be looked for among the weighted homogeneous functions $L(p, q)$ of weighted degree g_L equal to one of the KE $\bar{\rho}$. In particular, for $\bar{\rho}$ a positive multiple of $1/(d-1)$ this $L(p, q)$ can be a polynomial. The condition that all exponents are rational and that one of them is a positive multiple of $1/(d-1)$ is not expected to be a sufficient condition for integrability. Such a condition is provided by the Painlevé conjecture which we consider next.

According to the Painlevé conjecture [1] and its extension [2, 3] a sufficient condition for complete integrability is the Painlevé and the weak Painlevé property, respectively. A Hamiltonian has the Painlevé property if the only movable singularities of all the solutions of the equations of motion in the complex time plane are poles. In the case of the weak Painlevé property certain algebraic branch points also are allowed (see below). A strong necessary condition for the system having the Painlevé or weak Painlevé property is provided by an algorithm called singular point analysis or Painlevé analysis [1, 2]. Its main steps have been described in detail in [5]. Here we restrict ourselves to those aspects which are relevant in the comparison with Yoshida's theorems.

In the first step of Painlevé analysis the ansatz

$$q_i = c_i(t - t_0)^{-\mu_i} \qquad i = 1, 2 \tag{13}$$

is inserted into the equations of motion (5) where the general form of U and W defined by (6) and (7) is

$$U(q_1, q_2) = \sum_{k=0}^{d-1} u_k q_1^k q_2^{d-1-k} \tag{14}$$

$$W(q_1, q_2) = \sum_{k=0}^{2d} w_k q_1^k q_2^{2d-k} \tag{15}$$

where u_k, $k = 1, \ldots, d-1$ and w_k, $k = 1, \ldots, 2d$ are constants. We assume $U \not\equiv 0$ and $W \not\equiv 0$.

Expression (13) is supposed to be the leading term of a solution exhibiting a rational movable singularity at $t = t_0$. Consequently c_1 and c_2 are assumed to be non-zero and μ_1 and μ_2 to be rational numbers. For certain values of μ_i, $i = 1, 2$, some terms of the equations may balance while others can be ignored for $t \to t_0$. The former are called the dominant terms of the equations. In this way one obtains, in general, equations determining the amplitudes c_i, $i = 1, 2$. All possibilities of dominant behaviour specified by the sets $\{\mu_i, c_i, i = 1, 2\}$ must then be investigated further. The possibility $\mu_1 = \mu_2 = g$ for which all terms are dominant and the c_i satisfy (8) correspond to the special solutions considered in Yoshida's theorems. There, however, c_1 and c_2 are not restricted to be non-zero.

In the second step of Painlevé analysis

$$q_i = c_i(t-t_0)^{-\mu_i} + d_i(t-t_0)^{-\mu_i+r} \qquad i=1,2 \tag{16}$$

is inserted in the linearised form of the dominant terms. Resonances are those values of r for which the determinant of the linear system satisfied by d_i, $i=1,2$, vanishes. In the case $\mu_1=\mu_2=g$ this determinant is the Kowalevski determinant and the resonances are identical to the Kowalevski exponents. For other values of μ_i, $i=1,2$, this is not the case. A necessary condition for the system to have the Painlevé property is that all resonances are integers. In the case of the weak Painlevé property, resonances which are multiples of $1/s$, s being the common integer denominator of μ_1 and μ_2, are also allowed [2, 3, 5]. (We remark that the definition of resonance we use here differs by a factor s from the one used in [5].)

When the resonances satisfy these conditions, in the third step of Painlevé analysis it is tested whether the positive resonances indeed correspond to free parameters in a solution to the full equations of motion without logarithmic singularities. In the comparison with Yoshida's theorems only the first two steps of the Painlevé analysis play a role.

Let us consider now singularities (13) with $\mu_1<\mu_2=g$. Performing the first step of the Painlevé analysis one finds that these are possible when

$$w_0\neq 0,\ u_0=0,\ w_1=0 \text{ and } u_1w_2\neq 0 \tag{17}$$

the dominant terms in (5a) being linear in q_1 and the dominant terms in (5b) being independent of q_1. They balance provided

$$G_1(\mu_1;\, c_2)\equiv -\mu_1(\mu_1+1)-gu_1c_2^{d-1}-2w_2c_2^{2d-2}=0 \tag{18a}$$

$$G_2(c_2)\equiv -g(g+1)-2dw_0c_2^{2d-2}=0 \tag{18b}$$

which can be solved for μ_1 and c_2 in terms of u_1, w_0, w_2 and d. (Recall $g=1/(d-1)$.) The associated resonances are found to be $r=-1$, $2d/(d-1)$, 0 and $\bar r$ with

$$\bar r = 2\mu_1+1. \tag{19}$$

Considering on the other hand the amplitude equations (8) and the Kowalevski determinant (10) and (11) for solutions (3a) with $c_1=0$ one sees that

$$F_1(0,c_2)\equiv 0 \tag{20a}$$

$$F_2(0,c_2)\equiv G_2(c_2) \tag{20b}$$

$$M_{11}(\rho;0,c_2)\equiv G_1(g-\rho;\, c_2) \tag{20c}$$

and the Kowalevski exponents are (12) with

$$\bar\rho = 1/d-1-\mu_1 \tag{21}$$

$$= 1/d-1-\tfrac{1}{2}(\bar r-1). \tag{22}$$

This difference between resonances and Kowalevski exponents can be explained by the different choice of leading order in $t-t_0$ in equations (9) and (16). In an analogous way the relation between (16) with $\mu_2<\mu_1=g$ and (9) with $c_2=0$ can be established.

In general, still other Painlevé leading singularities having no counterpart in Yoshida's theorems might exist. These are not considered here. Instead we show how the results obtained so far can be used to combine the advantages of Painlevé analysis and Yoshida's theorems.

First, the theorems on Kowalevski exponents can be translated into theorems on resonances.

(i) If a second invariant of weighted degree g_L exists such that its gradient is not zero on a solution of (3) with c_1 and c_2 both non-zero, then there is a resonance $\bar{r} = g_L$ associated with the corresponding Painlevé leading singularity with $\mu_1 = \mu_2 = g$.

(ii) If a second invariant of weighted degree g_L exists such that its gradient is not zero on a solution of (3) with $c_1 = 0$, $c_2 \neq 0$ or $c_1 \neq 0$, $c_2 = 0$, then there is a resonance

$$\bar{r} = 2(g_L - 1/d - 1) + 1 \tag{23}$$

associated with the corresponding Painlevé leading singularity with $\mu_1 < \mu_2 = 1/(d-1)$ or $\mu_2 < \mu_1 = 1/(d-1)$ respectively.

(iii) A necessary condition for the existence of an algebraic second invariant is that all resonances with Painlevé leading singularities with $\mu_1 = 1/(d-1)$ or $\mu_2 = 1/(d-1)$ are rational numbers.

On the other hand, the restrictions imposed on the resonances by the Painlevé conjecture, namely that they should be multiples of $1/s$, s being the common integer denominator of μ_1 and μ_2, can be translated in the following restrictions on the Kowalevski exponents.

(*a*) All KE associated with solutions c_1, c_2 or (8) with c_1 and c_2 both non-zero must be multiples of $1/(d-1)$.

(*b*) All KE associated with solutions c_1, c_2 of (8) with $c_1 = 0$, $c_2 \neq 0$ (resp. $c_1 \neq 0$, $c_2 = 0$) must be multiples of $1/2s$, s being the integer common denominator of μ_1 (resp. μ_2) and $1/(d-1)$.

In a search for integrable systems Painlevé analysis, Yoshida's theorems and direct methods can be combined as follows.

First the KE are calculated. Next the restrictions coming from the second step of the Painlevé analysis are imposed and a direct search is performed for a polynomial second invariant of weighted degree equal to one of the KE. (We remark that it is a common procedure to do a search for a constant of the motion already after the second step of Painlevé analysis.) The calculation of the KE and the direct search for the polynomial invariants can be performed automatically using REDUCE packages developed by one of us [7, 8]. In the absence of success a second invariant can be looked for among non-polynomial functions, e.g. rational functions, or among functions of a different weight which then must have a gradient vanishing identically on the solution set of (4). According to the Painlevé conjecture this should only be pursued systematically when the system has also passed the third step of the Painlevé analysis.

As an illustrative example, we now consider the two-parameter class of Fokker–Planck Hamiltonians [5]

$$H(p, q) = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + p_1 q_1 q_2 + p_2(a q_1^2 + b q_2^2) \tag{24}$$

for which $d = 2$ and

$$U(q_1, q_2) = (1 - 2a)q_1 \tag{25a}$$

$$W(q_1, q_2) = -\tfrac{1}{2}a^2 q_1^4 - (\tfrac{1}{2} + ab)q_1^2 q_2^2 - \tfrac{1}{2}b^2 q_2^4. \tag{25b}$$

For $a = \frac{1}{2}$ this model is related by a gauge transformation to a one-parameter class of quartic potentials and is known to be integrable for $b = -1$, $b = -\frac{1}{2}$, 1 and 2. Here we assume $a \neq \frac{1}{2}$.

For $a \neq 0$, $b \neq 0$ and $b \neq -1/4a$ the solutions of equation (8) with the associated KE $\bar{\rho}$ are

(i) $\quad c_1 = \pm[(1-b)/a]^{1/2} \qquad c_2 = -1 \qquad \bar{\rho} = 1 + 2b \qquad$ (26)

(ii) $c_1 = \pm\{[(1+b)(3+\sqrt{ff'}) + 4ab(1-b)]/2af\}^{1/2}$

$c_2 = [f^{1/2} - f'^{1/2}]/(2f^{1/2})$

$\bar{\rho} = \frac{3}{2} + [\frac{9}{4} + f'b(2b+1)/f - (f'/f)^{1/2}(b-2)(2b+1)]^{1/2}2$ (27)

(iii) obtained from (ii) by replacing $f'^{1/2} \to -f'^{1/2}$

(iv) $c_1 = 0 \qquad c_2 = -1/b \qquad \bar{\rho} = 2 + 1/b$ (28)

(v) $c_1 = 0 \qquad c_2 = 1/b \qquad \bar{\rho} = \frac{3}{2} + (1/2b)[(b-2)^2 + 16ab]^{1/2}$ (29)

where

$$f = 1 + 4ab \tag{30}$$

$$f' = 9 + 4a(b-4). \tag{31}$$

We now impose the conditions coming from the second step of the Painlevé analysis. First, we exclude the special lines $b = 1$, $b = -\frac{1}{2}$ and $a = (2b-1)(b+1)/4b$. Then c_1 and c_2 are non-zero in cases (i)–(iii) and $c_1 = 0$, $c_2 \neq 0$ in cases (iv) and (v). Recalling that here $d = 2$ one has that a necessary condition for the weak Painlevé property to hold is that in cases (i)–(iii) $\bar{\rho}$ is integer and in cases (iv) and (v) $\bar{\rho}$ is rational. This is satisfied for the following values of a and b:

$b = 4 \qquad a = (18+k)(18-k)/(16k^2) \qquad k \in \mathbb{N}, k \neq 6$ (32a)

$b = 2 \qquad a = \frac{9}{8}, \frac{1}{8}, \frac{9}{128}, \frac{1}{32}, \frac{1}{392}$ (32b)

$b = \frac{1}{2} \qquad a = -\frac{1}{4}$ (32c)

$b = -2, \qquad a = \frac{3}{8}.$ (32d)

An interesting and finite subset can be singled out by the requirement that all integer KE are not bigger than a given integer n_0. Choosing $n_0 = 20$, cases (32a) for $k \geq 17$ are excluded. For the remaining 23 cases a direct search was performed. In three of them a polynomial second invariant was obtained with weighted degree equal to one of the KE. ((I)–(III) below). Proceeding in the same way for the exceptional lines $b = 1$, $b = -\frac{1}{2}$, $a = (2b-1)(b+1)/4b$ and $a = 0$, $b = 0$, $b = -1/4a$ two more integrals were found ((IV) and (V) below). Furthermore, applying the REDUCE package it was checked that no further polynomial first integrals exist of weighted degree ≤ 20.

In summary, the results obtained by a combined use of Painlevé analysis, Yoshida's theorems and direct methods is that the Hamiltonian (24) with $a \neq \frac{1}{2}$ is integrable in the following cases (the KE associated with alternatives (i)–(v) and the second invariant are given):

(I) $a = \frac{1}{8}$, $b = 2$

$\bar{\rho} = 5, 8, 8, \frac{5}{2}, 2$

$L(p,q) = p_1^4 + p_1^2 q_1(4p_1q_2 - \frac{3}{2}p_2q_1) + q_1^3(-\frac{1}{2}p_1^2q_1 - p_1p_2q_2 + \frac{1}{16}p_2^2q_1)$ (33)

$g_L = 8$

(II) $a = -\frac{1}{4}$, $b = \frac{1}{2}$

$\bar{\rho} = 2, 8, 5, 4, 4$

$L(p,q) = (2p_1^2 + p_2^2)(2p_2 + q_1^2 + 2q_2^2)^2$ (34)

$g_L = 8$

(III) $a = \frac{1}{32}$, $b = 2$

$$\bar{\rho} = 5, 10, 10, \tfrac{5}{2}, \tfrac{7}{4}$$

$$\begin{aligned} L(p, q) = 256p_1^4p_2 + 32p_1^2[4p_1^2(q_1^2 + 8q_2^2) + p_2^2q_1^2] \\ + q_1^2[64p_1^3q_1q_2 + 8p_1^2p_2(q_1^2 + 16q_2^2) + p_2^3q_1^2] \\ + q_1^4(p_1q_1 + 2p_2q_2)^2 \end{aligned} \tag{35}$$

$$g_L = 10$$

(IV) $a = 0, b = -1$

$$\bar{\rho} = 6, 3, 2$$

$$L(p, q) = p_1^2(2p_2 + q_1^2) \tag{36}$$

$$g_L = 6$$

(V) $a = 0, b = 1$, related to (IV) by a gauge transformation.

Except for case (II), which was obtained first by Hietarinta [13] applying a canonical transformation to an integrable velocity independent quartic potential, there results are new. In particular, we remark that case (III) is the first example ever given with a second invariant which is a polynomial of fifth order in the momenta.

More results will be reported elsewhere. In addition, the direct search program is being generalised to treat not only polynomial but also rational functions [8]. It can be expected that, encompassing this generalisation, a still closer relation will appear between properties at singular points and explicit determination of invariants.

Finally, we remark that the method is not restricted to two-dimensional systems and has been applied in the search for integrable three-dimensional quartic potentials [9].

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